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Magnetic hard squares: exact solution

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Abstract. A three-state IRF model of a magnetic hard-square lattice gas with anisotropic interactions is solved exactly on special two-dimensional manifolds in the full five-dimensional thermodynamic space spanned by the (ferro- or antiferro-) magnetic interactions J, K, the (attractive or repulsive) lattice-gas interactions L, M and the activity z. The phase boundaries determined include multicritical lines and critical surfaces as well as first-order surfaces of three- and fivefold phase coexistence between the fluid and various (para, ferro, etc) magnetic square-ordered solid phases. Analytic expressions are given for free energies, interfacial tensions, correlation lengths, sublattice densities and magnetisations. The critical behaviour of these quantities is studied and the associated critical exponents obtained.

In statistical mechanics, an interaction-round-a-face or IRF model (Baxter 1982a) is called exactly solvable (Tracy 1985) if it yields a parametrised family of solutions to the star-triangle or Yang-Baxter equations and hence possesses a family of commuting transfer matrices. It is now known (see, for example, Bazhanov and Stroganov 1982, Jimbo and Miwa 1985a) that there are many such solutions to the star-triangle equations and hence, in principle, many exactly solvable two-dimensional lattice models. However, to obtain all the thermodynamic quantities of interest such as the free energies, interfacial tensions, correlation lengths and order parameters (one-point functions) one needs to calculate the eigenvalue spectrum of both the row-to-row and corner transfer matrices. To date this program has only been accomplished (for a review see Pearce 1983) for the eight-vertex (Baxter 1972, 1973, 1982a, Johnson *et al* 1973) and hard-square (hexagon) models (Baxter 1980, Baxter and Pearce 1982, 1983, Pearce and Baxter 1984). In this paper I indicate briefly how this program is carried out for a three-state IRF model (Jimbo and Miwa 1985b) of a magnetic hard-square lattice gas and present the results. A full account of the calculations will be published elsewhere.

The magnetic hard-square model is a three-state IRF model that generalises and incorporates the magnetic Ising model and the hard-square (hexagon) lattice-gas models. To each site i of a square lattice assign a spin $\sigma_i = 0, \pm 1$; if the site is empty $\sigma_i = 0$, if the site is full $\sigma_i = +1$ or -1 as the spin of the particle is up or down respectively. The occupation number of site i is then $\sigma_i^2 = 0, 1$. The Boltzmann weight of a face (a, b, c, d being the four spins round a face, starting at the bottom-left and going anticlockwise) is taken to be

$$W(a,b,c,d) = \begin{cases} m(z_+t)^{(a^2+c^2)/4}(z_+/t)^{(b^2+d^2)/4}\exp(La^2c^2+Mb^2d^2+Jac+Kbd) & \\ \qquad\qquad ab=bc=cd=da=0 & \\ 0 \qquad\qquad \text{otherwise.} & \end{cases} \tag{1a}$$

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The activity $z_+ = z_-$ of the magnetic particles is shared out between adjacent faces of the lattice, L and M are diagonal (next-nearest neighbour) lattice-gas interactions, J and K are diagonal magnetic interactions, m is a trivial prefactor and t is a local parameter that cancels out of the partition function and row-to-row transfer matrix.

Subject to the nearest-neighbour exclusion $\sigma_i\sigma_j = 0$, there are seven independent Boltzmann weights:

$$\begin{aligned}
\omega_1 &= W(0,0,0,0) = m \\
\omega_2 &= W(\pm1,0,0,0) = W(0,0,\pm1,0) = m(z_+t)^{1/4} \\
\omega_3 &= W(0,\pm1,0,0) = W(0,0,0,\pm1) = m(z_+/t)^{1/4} \\
\omega_4 &= W(1,0,1,0) = W(-1,0,-1,0) = m(z_+t)^{1/2}\,\mathrm{e}^{L+J} \\
\omega_5 &= W(0,1,0,1) = W(0,-1,0,-1) = m(z_+/t)^{1/2}\,\mathrm{e}^{M+K} \\
\omega_6 &= W(1,0,-1,0) = W(-1,0,1,0) = m(z_+t)^{1/2}\,\mathrm{e}^{L-J} \\
\omega_7 &= W(0,1,0,-1) = W(0,-1,0,1) = m(z_+/t)^{1/2}\,\mathrm{e}^{M-K}.
\end{aligned} \qquad (1b)$$

These weights are invariant under spin reversal and reflection about a diagonal. Moreover, by reversing spins on alternate pairs of diagonals on a periodic lattice, we can assume without loss of generality that the magnetic interactions are ferromagnetic $(J, K \geq 0)$.

The Ising and hard-square (hexagon) models are limiting cases of the more general model (1). If m is set proportional to $(z_+\,\mathrm{e}^{L+M})^{-1/2}$ and the limit $z_+\,\mathrm{e}^{L+M} \to \infty$ is taken, with $L-M$ held constant, then two independent Ising models are obtained, one for each possible fully occupied sublattice. Likewise, if $J = K = 0$, the model reduces to the two-state hard-square (hexagon) models with the total activity of the particles given by

$$z = z_+ + z_- = 2z_+. \qquad (2)$$

Apart from the Ising limit, the magnetic hard-square model (1) can only be solved exactly on three special two-dimensional manifolds in the full five-dimensional space spanned by z, J, K, L and M. Let

$$\alpha = \tanh J \qquad \beta = \tanh K. \qquad (3)$$

Then these exact solution manifolds, which we denote by the letters H (generalised hard hexagon), E (elliptic) and T (trigonometric) are given by

$$\text{H} \qquad \begin{aligned} &\alpha = \beta = 0 \\ &z = 2z_+ = (1-\mathrm{e}^{-L})(1-\mathrm{e}^{-M})/(\mathrm{e}^{L+M} - \mathrm{e}^{L} - \mathrm{e}^{M}) \end{aligned} \qquad (4)$$

$$\text{E} \qquad \begin{aligned} &\mathrm{e}^{L} = (\alpha+\beta)/\beta(1-\alpha^2)^{1/2} \qquad \mathrm{e}^{M} = (\alpha+\beta)/\alpha(1-\beta^2)^{1/2} \\ &z = 2z_+ = \alpha\beta(1-\alpha^2)(1-\beta^2)(1+\alpha\beta)/(\alpha+\beta)^4 \end{aligned} \qquad (5)$$

$$\text{T} \qquad \begin{aligned} &\mathrm{e}^{L} = (1-\alpha^2)^{1/2}/\beta^2 \qquad \mathrm{e}^{M} = (1-\beta^2)^{1/2}/\alpha^2 \\ &z = 2z_+ = \alpha^2\beta^2. \end{aligned} \qquad (6)$$

For isotropic interactions ($\alpha = \beta$, $L = M$) the manifolds H, E and T reduce to the curves shown in figure 1.

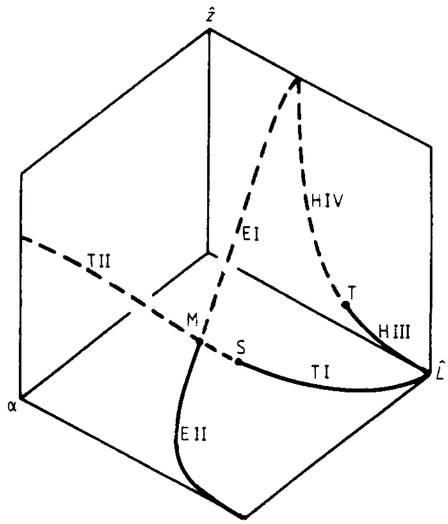

Figure 1. Exact solution curves H, E and T for the isotropic magnetic hard-square model showing the tricritical point T, the multicritical point M, the essential singularity S and the division into various regimes. For convenience in plotting the coordinates are α, $\hat{L} = e^L/(e^L+3)$ and $\hat{z} = 5z/(5z+2)$. These co-ordinates vary between 0 and 1.

The solution on the four physical regimes HI–HIV of the H manifold has been presented in detail in previous papers. We will therefore only present the results for the E and T manifolds. To do so we need the following standard elliptic functions of name q ($|q|<1$):

$$\begin{aligned} \theta_1(u, q) &= 2q^{1/4} \sin u \prod_{n=1}^{\infty} (1-2q^{2n}\cos 2u + q^{4n})(1-q^{2n}) \\ \theta_4(u, q) &= \prod_{n=1}^{\infty} (1-2q^{2n-1}\cos 2u + q^{4n-2})(1-q^{2n}) \\ k'(q) &= \prod_{n=1}^{\infty} [(1-q^{2n-1})/(1+q^{2n-1})]^4 \\ Q(q) &= \prod_{n=1}^{\infty} (1-q^n). \end{aligned} \tag{7}$$

The E manifold

The E manifold (5) admits a natural elliptic parametrisation

$$\begin{aligned} \alpha &= \theta_1(\pi/3-u, q)\theta_4(\pi/3, q)/\theta_1(\pi/3, q)\theta_4(\pi/3-u, q) \\ \beta &= \theta_1(u, q)\theta_4(\pi/3, q)/\theta_1(\pi/3, q)\theta_4(u, q) \end{aligned} \tag{8a}$$

where $0 \leq u \leq \pi/3$ and the nome q is given by

$$(1-\alpha^2)(1-\beta^2)/\alpha\beta(1+\alpha\beta) = [\theta_4(0, q)/\theta_4(\pi/3, q)]^3 \equiv \mu^3. \tag{8b}$$

The curves (8b) of constant q fill the E manifold (5). The special curve $q=0$ is a line of multicritical points which separates the surface into two distinct physical regimes which we denote by EI ($q<0$) and EII ($q>0$).

EI ($q<0$): paramagnetic square-ordered solid

Throughout regimes EI and EII the partition function per site is given by

$$\kappa/\omega_1=(\omega_4\omega_5+\omega_6\omega_7)/(\omega_4\omega_5-\omega_6\omega_7)$$
$$=\theta_1(\pi/3,q)/\theta_1(u+\pi/3,q)=\cosh(J+K). \tag{9}$$

The free energy varies analytically on E even across the line of multicritical points. At first sight this seems surprising, but presumably the free energy is *not* analytic on the multicritical line when considered as a function over the full five-dimensional thermodynamic space. Certainly, the correlation lengths diverge as the multicritical line is approached. There are in fact two different correlation lengths: ξ_m corresponding to the decay of magnetisation-magnetisation correlations $\langle\sigma_i\sigma_j\rangle$ and ξ_ρ corresponding to the decay of density-density correlations $\langle\sigma_i^2\sigma_j^2\rangle$. In regime EI the correlation lengths and interfacial tension σ are given by

$$2\xi_m^{-1}=\xi_\rho^{-1}=2\beta\sigma=-\ln k'(|q|^{3/2}) \tag{10}$$

where $\beta=1/k_BT$ is the inverse temperature. Taking q as the deviation from criticality variable we see that the associated critical exponents are $\nu=\mu=\frac{3}{2}$.

In regime EI one sublattice is occupied preferentially over the other. Let ρ_1 and ρ_2 be the two sublattice densities and suppose $\rho_1>\rho_2$. Then the order parameter is

$$R=\rho_1-\rho_2=(\sqrt{8}/3)|q|^{1/8}Q(q)Q(q^2)Q^2(q^3)Q^3(q^{12})/Q^7(q^6) \tag{11}$$

which clearly vanishes as the multicritical line is approached ($q\to0-$) with an exponent $\beta_1=\frac{1}{8}$. The mean total density of the solid

$$\rho=\frac{1}{2}(\rho_1+\rho_2)=\frac{1}{3}\frac{Q^2(q)Q(q^2)}{Q(q^4)}\frac{Q^2(q^3)Q^3(q^{12})}{Q^7(q^6)}\equiv D(q) \tag{12}$$

on the other hand, assumes the value $\rho_c=\frac{1}{3}$ as $q\to0-$. The sublattice magnetisations are $m_1=m_2=0$, so regime EI lies entirely within the paramagnetic square-ordered solid phase.

EII ($q>0$): fluid and ferromagnetic square-ordered solid

Regime EII is a first-order surface of five-fold coexistence between the fluid and four ferromagnetic square-ordered solid phases. The partition function per site is given by (9) and the correlation lengths and interfacial tension by

$$\xi_m^{-1}=\xi_\rho^{-1}=2\beta\sigma=-\ln k'(q^{3/2}) \tag{13}$$

which is strikingly similar to (10) and leads to the same exponents $\nu=\mu=\frac{3}{2}$. Although several interfaces are possible, they all have the same interfacial tension σ as given by (13).

In the solid phase the sublattice density difference is

$$R=\rho_1-\rho_2=\tfrac{4}{3}q^{1/8}Q(q)Q(q^2)\frac{Q(q^6)}{Q^2(q^{3/2})Q(q^3)}. \tag{14}$$

This order parameter vanishes at the multicritical line with an exponent $\beta_1=\frac{1}{8}$ as in EI. The mean total density of the solid is

$$\rho^{\text{solid}}=\frac{1}{2}(\rho_1+\rho_2)=\frac{1}{3}\frac{Q^7(q)}{Q^2(q^{1/2})Q^3(q^2)}\frac{Q(q^6)}{Q^2(q^{3/2})Q(q^3)}=D(-q^{1/2}) \tag{15}$$

compared with the fluid density

$$\rho^{\text{fluid}}=\frac{1}{3}\frac{Q^2(q^{1/2})Q(q)}{Q(q^2)}\frac{Q^2(q^{3/2})Q^3(q^6)}{Q^7(q^3)}=D(q^{1/2}). \tag{16}$$

Both these densities have the multicritical value $\rho_c=\frac{1}{3}$ and the density difference $\Delta\rho=\rho^{\text{solid}}-\rho^{\text{fluid}}$ vanishes with an exponent $\beta_2=\frac{1}{2}$. The sublattice magnetisations in the solid phase are

$$m_1=(\tfrac{8}{3})^{1/2}q^{3/16}\frac{Q^2(q)}{Q(q^2)}\frac{Q(q^3)Q(q^6)}{Q^3(q^{3/2})} \qquad m_2=0 \tag{17}$$

so the magnetisation $m=\frac{1}{2}(m_1+m_2)$ vanishes at the multicritical line with exponent $\beta=\frac{3}{16}$. There is no magnetisation in the fluid phase, that is, $m^{\text{fluid}}=0$. The fluid is therefore paramagnetic. Indeed, there are no indications at all of the existence of a magnetic fluid for this model.

The T manifold

The T manifold (6) divides into two physical regimes which are naturally parametrised in terms of trigonometric (TI) or hyperbolic (TII) functions by the substitutions

$$\alpha=s(\lambda-u)/s(\lambda) \qquad \beta=s(u)/s(\lambda) \tag{18a}$$

where $0<u<\lambda$ and $s(u)=\sin u$ (TI) or $\sinh u$ (TII). It follows that

$$\Delta=(1-\alpha^2-\beta^2)/2\alpha\beta=c(\lambda) \tag{18b}$$

where $c(\lambda)=\cos\lambda$ (TI) or $\cosh\lambda$ (TII). The curves (18*b*) of constant Δ or λ fill the T manifold (6). In this instance, the special curve $\Delta=1$ or $\lambda=0$ separating regimes TI and TII is a line of essential singularities.

TI ($\Delta>1$): fluid and paramagnetic square-ordered solid

Regimes TI and TII have many similarities to the six-vertex model. Indeed the partition function per site κ satisfies the same inversion relations (Baxter 1982b). Subject to some analyticity assumptions, the free energies are therefore given by the six-vertex expressions. In TI

$$\ln(\kappa/\omega_1)=2\sum_{n=1}^{\infty}\frac{\exp(-2n\lambda)\sinh nu\sinh n(\lambda-u)}{n\cosh n\lambda}. \tag{19}$$

This regime is in fact a first-order surface of three-fold coexistence between the fluid and the two paramagnetic square-ordered solid phases. The sublattice density difference, solid and fluid densities are respectively

$$R=\rho_1-\rho_2=(2/\sqrt{3})s^{1/4}Q(s^6)Q(s^{18})/Q^2(s^9) \tag{20}$$

$$\rho^{\text{solid}}=\frac{1}{2}(\rho_1+\rho_2)=\frac{1}{3}\frac{Q^2(s^2)Q(s^3)}{Q(s)Q(s^6)}\frac{Q(s^{18})}{Q^2(s^9)} \tag{21}$$

$$\rho^{\text{fluid}}=\frac{1}{3}\frac{Q^2(s)Q(s^{18})}{Q(s^2)Q^2(s^9)} \tag{22}$$

where

$$s=\exp(-\pi^2/9\lambda). \tag{23}$$

At criticality $\lambda = s = 0$, $R = 0$ and $\rho^{\text{solid}} = \rho^{\text{fluid}} = \rho_c = \frac{1}{3}$. Taking λ as the natural deviation from criticality variable, we see from (23) that R and $\Delta\rho = \rho^{\text{solid}} - \rho^{\text{fluid}}$ vanish with essential singularities and not the usual simple power laws. The fluid and solid phases are paramagnetic, that is, $m_1 = m_2 = m^{\text{fluid}} = 0$. The correlation lengths and interfacial tension have not been obtained in this regime or TII.

TII ($|\Delta| < 1$): critical surface

The free energy in regime TII is given by the six-vertex expression

$$\ln(\kappa/\omega_1) = \int_{-\infty}^{\infty} \mathrm{d}t \frac{\cosh(\pi - 2\lambda)t \sinh ut \sinh(\lambda - u)t}{t \sinh \pi t \cosh \lambda t}. \tag{24}$$

Unlike the other regimes, TII does not exhibit a limit of extreme order or disorder. In analogy with the six-vertex model, we therefore expect TII to be a critical surface with $\xi^{-1} = \beta\sigma = R = m_1 = m_2 = 0$. Consistent with this claim we observe that TII intersects the E manifold precisely along the multicritical line which is the curve corresponding to $\lambda = \pi/3$ or $\Delta = \frac{1}{2}$. Likewise, evaluation of the integral in (24) for $\lambda = \pi/3$ gives agreement with the result (9), increasing one's confidence that the analyticity assumptions (Baxter 1982b) leading to (24) are correct.

Transfer matrix equation: E manifold

The row-to-row transfer matrix is defined by

$$V_{\sigma,\sigma'} = \prod_{j=1}^{N} W(\sigma_j, \sigma_{j+1}, \sigma'_{j+1}, \sigma'_j) \tag{25}$$

where the face weights are given by (1) and $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ are the configurations of two successive periodic rows of N spins. The parametrisations (8) with q fixed, and (18) with λ fixed, each give a one-parameter family $\mathbf{V}(u)$ of commuting transfer matrices. This has been established by Jimbo and Miwa (1985b) who showed that both the E and T families satisfy the star-triangle equations. We show further that the E family satisfies a remarkable functional equation. Although it seems likely that the T family satisfies a functional equation similar to that of the six-vertex model, such a functional equation has not been found.

Let

$$\mathbf{T}(u) = [(\omega_4\omega_5 - \omega_6\omega_7)/\omega_1(\omega_4\omega_5 + \omega_6\omega_7)]^N \mathbf{RV}(u) \tag{26}$$

where the spin-reversal operator is

$$R_{\sigma,\sigma'} = \prod_{j=1}^{N} \delta(\sigma_j, -\sigma'_j). \tag{27}$$

Then the E family of matrices $\mathbf{T}(u)$ satisfies

$$\mathbf{T}(u)\mathbf{T}(u+\pi/3)\mathbf{T}(u+2\pi/3) = \mathbf{I} + (-1)^N\mathbf{R} + \mathbf{T}(u) + \mathbf{T}(u+\pi/3) + \mathbf{T}(u+2\pi/3) \tag{28a}$$

$$\mathbf{T}(u+\pi) = (-1)^N \mathbf{RT}(u) \tag{28b}$$

where $\mathbf{I}$ is the identity matrix. The derivation of this functional equation is involved and will only be sketched here. Let $\mathbf{V}=\mathbf{V}(u)$, $\mathbf{V}'=\mathbf{V}(u+\pi/3)$, $\mathbf{V}''=\mathbf{V}(u+2\pi/3)$ and similarly for the face weights W, W' and W''. Then from (25) it follows that

$$[\mathbf{V}\mathbf{V}'\mathbf{V}'']_{\sigma,\sigma'}=\operatorname{Tr}\mathbf{S}(\sigma_1,\sigma_2,\sigma_2',\sigma_1')\mathbf{S}(\sigma_2,\sigma_3,\sigma_3',\sigma_2')\ldots\mathbf{S}(\sigma_N,\sigma_1,\sigma_1',\sigma_N') \tag{29}$$

where the 25 $\mathbf{S}$ matrices have elements

$$[\mathbf{S}(\sigma_1,\sigma_2,\sigma_2',\sigma_1')]_{\tau_1,\tau_2;\tau_1',\tau_2'}=W(\sigma_1,\sigma_2,\tau_2,\tau_1)\,W'(\tau_1,\tau_2,\tau_2',\tau_1')\,W''(\tau_1',\tau_2',\sigma_2',\sigma_1'). \tag{30}$$

In general each $\mathbf{S}$ matrix in (29) is a 5×5 matrix. If, however, $\sigma_j^2=(\sigma_j')^2=1$ then by exclusion $\tau_j=\tau_j'=0$ and the corresponding $\mathbf{S}$ matrices have one row or column. There are eight such $\mathbf{S}$ matrices. It turns out that each of these is a linear combination of some simple (u-independent) left or right eigenvectors of the full 5×5 matrix $\mathbf{S}(0,0,0,0)$. By considering the action of the other $\mathbf{S}$ matrices on these eigenvectors and using their orthogonality properties combined with the periodic boundary conditions, it can be shown that the non-zero elements of $\mathbf{V}\mathbf{V}'\mathbf{V}''$ fall into three categories: either (i) $\sigma_j=\sigma_j'$ for all j, (ii) $\sigma_j=-\sigma_j'$ for all j or (iii) $\sigma_j\sigma_j'=0$ for all j where, for convenience, the case $\sigma_j=\sigma_j'=0$ for all j is included in category (iii). For matrix elements falling into categories (i) and (ii) it is now straightforward to verify that they are of the form required by (28a). For matrix elements in category (iii), however, it is necessary to effect the simultaneous triangularisation of the remaining seventeen 5×5 $\mathbf{S}$ matrices; the trace in (29) can then be evaluated immediately from the diagonal elements. Although this task appears daunting the number of independent 5×5 matrices is reduced from seventeen to ten by using spin-reversal symmetry. The triangularisation proceeds by making proper use of the known simple eigenvectors. In this way (28a) is eventually verified for all allowed matrix elements.

To solve the functional equations (28) for the eigenvalues of the row-to-row transfer matrix it is simplest to go over to a conjugate modulus parametrisation. Define x and w by

$$|q|=\exp(-\varepsilon)\qquad x=\exp(-\pi^2/3\varepsilon)\qquad w=\exp(-2\pi u/\varepsilon) \tag{31}$$

so that $0<x<1$ and $x^2<w<1$ throughout the physical regimes. Then a convenient parametrisation to use is:

EI
$$\begin{aligned}
&\mu=f(-x^3)/x^{1/3}f(-x) && \omega_1=f(-xw^{-1})/f(-x)\\
&\omega_2=(\mu/2)^{1/2}x^{2/3}f(w)/wf(x) && \omega_3=f(x^2w^{-1})/f(x^2)\\
&\omega_4-\omega_6=w^{-1/2}f(x^2w^{-1})/f(x^2) && \omega_4+\omega_6=w^{-1}f(-xw)/f(-x)\\
&\omega_5-\omega_7=2\mu^{-1}x^{2/3}w^{-1/2}f(w)/f(x^2) && \omega_5+\omega_7=2f(-x^3w)/f(-x^3)
\end{aligned}\tag{32}$$

EII
$$\begin{aligned}
&\mu=x^{2/3}f(-1)/f(-x^2) && \omega_1=f(-x^2w)/f(-x^2)\\
&\omega_2=(\mu/2)^{1/2}x^{2/3}f(w)/f(x^2) && \omega_3=f(x^2w^{-1})/f(x^2)\\
&\omega_4-\omega_6=wf(x^2w^{-1})/f(x^2) && \omega_4+\omega_6=wf(-x^2w^{-1})/f(-x^2)\\
&\omega_5-\omega_7=2\mu^{-1}x^{2/3}f(w)/wf(x^2) && \omega_5+\omega_7=2f(-w)wf(-1).
\end{aligned}\tag{33}$$

Here $f(w)\equiv f(w,x^6)$ with

$$f(w,x)=\prod_{n-1}^{\infty}(1-x^{n-1}w)(1-x^nw^{-1})(1-x^n). \tag{34}$$

With this parametrisation the functional equations (28) become

$$\mathbf{T}(w)\mathbf{T}(x^2w)\mathbf{T}(x^4w)=\mathbf{I}+(-1)^N\mathbf{R}+\mathbf{T}(w)+(-1)^N\mathbf{R}\mathbf{T}(x^2w)+\mathbf{T}(x^4w) \tag{35a}$$

$$\mathbf{T}(x^6w)=(-1)^N\mathbf{R}\mathbf{T}(w). \tag{35b}$$

The next step is to use the analyticity and periodicity properties of (32) and (33) in the complex w plane. Since the eigenvectors must be independent of w, it can be shown that the eigenvalues of $\mathbf{V}(w)$ and $\mathbf{T}(w)$ must be of the form:

EI

$$\begin{gathered} V(w)=\tau w^{p/2}\prod_{j=1}^{N}[f(w/w_j)/f(1/w_j)] \\ T(w)=[w^{1/2}f(x^2)/f(x^2w)]^N RV(w) \\ \prod_{j=1}^{N} w_j=(-1)^N Rx^{-3p-5N} \end{gathered} \tag{36}$$

EII

$$\begin{gathered} V(w)=\tau w^{n}\prod_{j=1}^{N}[f(w/w_j)/f(1/w_j)] \\ T(w)=[f(x^2)/f(x^2w)]^N RV(w) \\ \prod_{j=1}^{N} w_j=(-1)^N Rx^{-6n-2N}. \end{gathered} \tag{37}$$

Here $R=\pm 1$ is the eigenvalue of $\mathbf{R}$, n and p are integers with $p=\frac{1}{2}(1+R)\pmod 2$, $\tau=V(1)$ is an nth root of unity (since $\mathbf{V}(1)$ is the shift operator) and $w_1, w_2, \ldots, w_N$ are the N zeros of $V(w)$ within a period annulus.

Eigenvalue spectrum: EI and EII

The eigenvalue spectrum of the row-to-row transfer matrix $\mathbf{T}(w)$ in regimes EI and EII breaks up into a series of regular eigenvalues, plus some additional exceptional eigenvalues. The regular eigenvalues are characterised by the behaviour

$$|T(w)|=\begin{cases} O(1) & x^3<|w|<x, \qquad x<|w|<x^{-1} \\ O(x^{-\varepsilon N}) & x^5<|w|<x^4 \qquad x^4<|w|<x^3. \end{cases} \tag{38}$$

The precise form of these eigenvalues is

$$T_{p,r}(w)=Kw^{p/2}f^{\nu}(x^2w/a_0)\prod_{j=1}^{(N-p-\nu)/2} f(xw/a_j)f(x^3w/a_j')\prod_{k=1}^{p} f(x^5w/b_k)/f^N(x^2w) \tag{39}$$

where a_0, a_j, a_j', b_k are complex numbers, $p=N-\nu\pmod 2$, $\nu=\frac{1}{2}(1+R)$ in EI, $\nu=\frac{1}{2}[1+(-1)^N]$ in EII and the constant K is determined by the requirement $T(1)=R\tau$. The regular eigenvalues occur in bands labelled by p and R, with $p=0,1,2,\ldots$ if N is even and $p=1,2,3\ldots$ if N is odd. Using (38), and neglecting exponentially small terms in (35a) for large N, we find that the regular eigenvalues must satisfy the simple functional equation

$$T(w)T(x^2w)=1 \qquad x<|w|<x^{-1}. \tag{40}$$

Solving this for $T(w)$ of the form (39) gives

$$T(w) = \prod_{k=1}^{p} \psi(w/b_k) \qquad x^3 < |w| < x^{-1} \tag{41a}$$

$$\psi(w) = \mathrm{i} w^{1/2} f(x^3 w, x^4)/f(xw, x^4). \tag{41b}$$

To obtain equations for the complex numbers b_k involves solving for $T(w)$ outside the annulus $x^3 < |w| < x^{-1}$. These calculations, which are too lengthy to give here, show that for the bands of largest eigenvalues of interest the b_k are unimodular and dense on the unit circle for large N.

The exceptional eigenvalues have the form ($R = \pm 1$):

$$T_{X,R}(w) = K \prod_{j=1}^{N} f(x^2 w/a_j)/f^N(x^2 w) \tag{42a}$$

$$|T_{X,R}(w)| = \mathrm{O}(1) \qquad x^5 < |w| < x^{-1}. \tag{42b}$$

In the thermodynamic limit ($N \to \infty$), it is found that

$$T_{X,R}(w) = \pm 1. \tag{43}$$

Clearly these eigenvalues belong with the largest ($p = 0$, $R = 1$) band of regular eigenvalues $T_{0,1}(w) = 1$. For N even, the total number of largest eigenvalues corresponds to the number of coexisting phases. In EI there are two largest eigenvalues: $T_{0,1} = 1$, $T_{X,1} = -1$. In EII there are five largest eigenvalues: two regular eigenvalues $T_{0,1} = 1$ and three exceptional eigenvalues $T_{X,1} = -1$ and $T_{X,-1} = \pm 1$. The interfacial tension σ can be calculated from the asymptotic degeneracy of these eigenvalues and the correlation lengths obtained from the gap to the appropriate band of next-largest eigenvalues.

Sublattice densities and magnetisations

The sublattice densities and magnetisations in regimes EI, EII and TI are obtained by using corner transfer matrices (Baxter 1981a, b, 1982a). The recursion relations that arise can be solved using Gaussian polynomials as was first done by Andrews (1981). The results (14)-(17) and (22) were essentially obtained by Jimbo and Miwa (1985b). The results (11), (12), (20) and (21), however, are new and the working, which is similar to the other cases, will be given elsewhere. The corner transfer matrix methods fail in regime TII. However, if this is a critical surface as seems likely, then $R = m_1 = m_2 = 0$. Finally, since $\rho_1 = \rho_2 = \rho_c = \frac{1}{3}$ on the multicritical line and on the line of essential singularities, it is tempting to conjecture that $\rho_1 = \rho_2 = \frac{1}{3}$ throughout regime TII.

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